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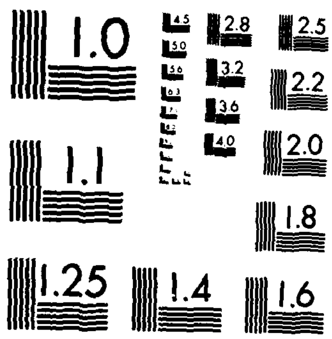

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ON THE FINITE ELEMENT APPROXIMATION OF THE STREAMFUNCTION-VORTICITY EQUATIONS

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Finite element algorithms are presented for the approximate solution of the streamfunction-vorticity equations of steady incompressible viscous flows. Both the linear Stokes and the nonlinear Navier-Stokes equations are considered. The methods discussed require low continuity finite element spaces and do not require any artificial specification of the vorticity at solid boundaries. Particular attention is paid to methods for multiply connected domains and to theoretical and computational estimates for the accuracy of the algorithms. Brief consideration is also given to three dimensional problems, to exterior problems, and to the recovery of the pressure field.

I - Formulations

The stationary Navier-Stokes equations, written in terms of the streamfunction ψ and the vorticity ω, are given by

$$\Delta\psi = -\omega \quad \text{in } \Omega \tag{1}$$

and

$$-\nu\Delta\omega + \left(\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}\right) = \text{curl } \underline{f} \quad \text{in } \Omega \tag{2}$$

where in (1) and (2) ν is the kinematic viscosity, $\underline{f}$ the body force, and Ω is a bounded region in R^2. If Ω is multiply connected, we denote by Γ_o its exterior boundary and by Γ_i, $i = 1,\ldots,m$, the remaining parts of the boundary. See Figure 1.

Suppose that at these boundaries the velocity $\underline{u}$ is specified, i.e., $\underline{u} = \underline{g}$ on $\Gamma = \cup_{i=0}^{m} \Gamma_i$. In order for a streamfunction to exist, we must have that [1]

$$\int_{\Gamma_i} \underline{g}\cdot\underline{n}\, d\sigma = 0 \quad \text{for } i = 0,\ldots,m, \tag{3}$$

i.e., there is no net mass flow through any of the boundary pieces. It is well known that the streamfunction is unique up to an additive constant, and we fix its value by specifying it to be zero at an arbitrary point x_o on Γ_o. Now, let q denote a function such that

$$\frac{\partial q}{\partial \tau} = \underline{g}\cdot\underline{n} \quad \text{on } \Gamma \quad \text{and} \quad q(x_o) = 0. \tag{4}$$

Then the boundary conditions for the system (1) and (2) are given by

$$\psi = q \text{ on } \Gamma_o,\ \psi = q + a_i \text{ on } \Gamma_i,\ i = 1,\ldots,m \tag{5}$$

and

$$\frac{\partial\psi}{\partial n} = -\underline{g}\cdot\underline{\tau} \quad \text{on } \Gamma, \tag{6}$$

where a_i, $i = 1,\ldots,m$, are constants to be determined as part of the solution. These constants are fixed by the requirement that the pressure be single valued, i.e., the change in the pressure p around any closed curve surrounding any of Γ_i, $i = 1,\ldots,m$, is zero. Since the momentum equation stipulates that

$$\frac{1}{\rho}\nabla p = -\underline{u}\cdot\nabla\underline{u} + \nu\Delta\underline{u} + \underline{f} \tag{7}$$

where ρ is the constant density, we have that, for $i = 1,\ldots,m$,

$$0 = \frac{1}{\rho}\int_{\Gamma_i} \nabla p\cdot\underline{\tau}\, d\sigma = \int_{\Gamma_i} (\nu\Delta\underline{u} - \underline{u}\cdot\nabla\underline{u} + \underline{f})\cdot\underline{\tau}\, d\sigma.$$

Since $\Delta\underline{u}\cdot\tau = \partial\omega/\partial n$ whenever $\text{div } \underline{u} = 0$ and $\underline{u}\cdot\nabla\underline{u}\cdot\underline{\tau} = \omega\underline{u}\cdot\underline{n} + \partial[(\underline{u}\cdot\underline{u})/2]/\partial\tau$, we then have that

$$\int_{\Gamma_i} \left(\nu\frac{\partial\omega}{\partial n} - \omega\underline{g}\cdot\underline{n} + \underline{f}\cdot\underline{\tau}\right) d\sigma = 0 \quad \text{for } i = 1,\ldots,m. \tag{8}$$

Thus (1,2,5,6, and 8) are the governing equations and side conditions which are to determine the functions ψ and ω and the constants a_i, $i = 1,\ldots,m$.

Weak Formulation

We define the function spaces

$$H^r(\Omega) = \{\phi \in L^2(\Omega);\ \frac{\partial^p\phi}{\partial x^\alpha \partial y^\beta} \in L^2(\Omega),\ \alpha,\beta,p,r \in Z^+,$$

$$\alpha+\beta = p,\ p = 0,\ldots,r\},$$

$$H_o^1(\Omega) = \{\phi \in H^1(\Omega);\ \phi = 0 \text{ on } \Gamma\},$$

$$H_a^1(\Omega) = \{\phi \in H^1(\Omega);\ \phi = 0 \text{ on } \Gamma_o;\ \phi = c_i \text{ on } \Gamma_i,$$

$$i = 1,\ldots,m,\ c_i \text{ arbitrary}\},$$

$$W^{1,4}(\Omega) = \{\phi \in L^4(\Omega);\ \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \in L^4(\Omega)\},$$

$$V_a = H_a^1(\Omega) \cap W^{1,4}(\Omega), \quad V = H^1(\Omega) \cap W^{1,4}(\Omega),$$

and the set

$$S_a = \{\phi \in H^1(\Omega) \cap W^{1,4}(\Omega);\ \phi = q \text{ on } \Gamma_o;$$

$$\phi = q + c_i \text{ on } \Gamma_i,\ i = 1,\ldots,m,\ c_i \text{ arbitrary}\}.$$

In addition we will make use of the spaces $H^{r-\frac{1}{2}}(\Gamma)$ consisting of traces of functions belonging to $H^r(\Omega)$ for $r \in Z^+$, and also the dual spaces with negative indices of the above defined spaces. For details concerning these spaces, see [1,2].

The weak formulation of the streamfunction-vorticity problem which we will utilize is as follows.

Given $\underline{f} \in [L^2(\Omega)]^2$ and $\underline{g} \in [H^{-\frac{1}{2}}(\Gamma)]^2$ satisfying (3), we seek $\psi \in S_a$ and $\omega \in V$ such that

$$\int_\Omega \omega\zeta\, d\Omega - \int_\Omega \underline{\text{curl}}\ \psi\cdot\underline{\text{curl}}\ \zeta\, d\Omega = \int_\Gamma \zeta\underline{g}\cdot\underline{\tau}\, d\sigma \tag{9}$$

for all $\zeta \in V$,

$$-\nu\int_\Omega \underline{\text{curl}}\ \omega\cdot\underline{\text{curl}}\ \phi\ d\Omega + \int_\Omega \omega\left(\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}\right)d\Omega \tag{10}$$

$$= -\int_\Omega \underline{f}\cdot\underline{\text{curl}}\ \phi\ d\Omega \quad \text{for all} \quad \phi\ \varepsilon\ V_a$$

where $\underline{\text{curl}} = (\partial/\partial y, -\partial/\partial x)$. This weak formulation is the most practical special case of a more general formulation found in [1,3]. Also, this formulation was used in [4] to successfully compute high Reynolds number driven cavity flows on nonuniform, relatively coarse, grids. We note at the outset that the only boundary conditions explicitly imposed on the functions appearing in (9) and (10) are those corresponding to (5). In particular, (6) is a natural boundary condition and no boundary conditions on ω are imposed on boundaries where (5) and (6) apply. Furthermore, the constraints (8) are also natural to the formulation (9-10). To elucidate these points and to show the connection between (9-10) and (1,2,5,6, and 8), let us proceed formally and perform appropriate integrations by parts in (9-10). We are then led to

$$\int_\Omega (\omega+\Delta\psi)\zeta\ d\Omega = \int_\Gamma \left(\frac{\partial\psi}{\partial n}+\underline{g}\cdot\underline{\tau}\right)\zeta\ d\sigma, \tag{11}$$

and

$$\int_\Omega \left(\nu\Delta\omega - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}+\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}+\text{curl}\ \underline{f}\right)\phi\ d\Omega \tag{12}$$

$$= \int_\Gamma \left(\nu\frac{\partial\omega}{\partial n} - \omega\phi\frac{\partial\psi}{\partial\tau}+\underline{f}\cdot\underline{\tau}\right)\phi\ d\sigma.$$

Since we may choose ζ to alternately vanish and not vanish on the boundary Γ, we recover, from (11), (1) and (6). Also (2) is recovered from (12) by choosing ϕ to vanish on the boundary Γ. Since ϕ is required to vanish on Γ_o, the integral on the right hand side of (12) vanishes on Γ_o. By alternately choosing ϕ to vanish on all Γ_i, $i = 1,\ldots,m$, except on one, say Γ_j, on which ϕ is required to be a constant, that same integral yields, in view of (4) and (5), that (8) is satisfied. Then indeed the side conditions (6) and (8) are natural to the formulation (9-10), and the only essential boundary conditions are those on ψ itself, i.e., (5). Again we note that no boundary condition on ω need be imposed.

In addition to the ease with which the boundary conditions are satisfied, the formulation (9-10) also allows for the use of low continuity, i.e., merely continuous over Ω, function spaces and thus greatly simplifies its finite element discretization. Also, as a practical observation, we note that the function q is required only on the boundaries Γ_i, $i = 0,\ldots,m$, and thus usually may be easily computed from (4) separately on each of these parts of the boundary.

So far we have only considered boundary conditions which correspond to a specification of the velocity at all boundaries and we will continue to focus on this case in the sequel. However, by examining (11-12), we see that many other kinds of boundary conditions can be also implemented. For example, suppose $\underline{f} = 0$ and that $\omega = \partial\omega/\partial n = 0$ on part of the boundary, say Γ_o. Such a boundary condition is useful in matching a viscous flow to an external inviscid flow. In this case we retain (9-10) with $\underline{f} = 0$ but now $\omega,\zeta\ \varepsilon\ H^1(\Omega)$ are required to vanish on Γ_o and ψ, ϕ are not specified there. With $\omega = 0$ on Γ_o and ϕ arbitrary there, we see that the right hand side of (12) yields $\partial\omega/\partial n = 0$ as a natural boundary condition.

The Linear Case

We will also be interested in the linear Stokes flow case, and it will be advantageous to sometimes consider this case separately. For this case, one simply omits all terms arising from the nonlinear convection terms in (2). Thus (2) is replaced by $\Delta\omega = -\text{curl}\ \underline{f}$ where we have absorbed the constant ν into $\underline{f}$. Also, the weak formulation (9-10) is replaced by the following. We seek $\omega\ \varepsilon\ H^1(\Omega)$ and $\psi\ \varepsilon\ \{\phi\ \varepsilon\ H^1(\Omega);\ \phi = q \text{ on } \Gamma_o;\ \phi = q+c_i \text{ on } \Gamma_i,\ i = 1,\ldots,m,\ c_i \text{ arbitrary}\}$ such that (9) and

$$\int_\Omega \underline{\text{curl}}\ \omega\cdot\underline{\text{curl}}\ \phi\ d\Omega = \int_\Omega \underline{f}\cdot\underline{\text{curl}}\ \phi\ d\Omega \tag{13}$$

$$\text{for all} \quad \phi\ \varepsilon\ H^1_a(\Omega)$$

are satisfied. Note that it is no longer required for $\phi,\psi\ \varepsilon\ W^{1,4}(\Omega)$ since this inclusion results from the need to make the nonlinear term in (10) well defined. Also, (8) is replaced by

$$\int_{\Gamma_i} \left(\frac{\partial\omega}{\partial n}+\underline{f}\cdot\underline{\tau}\right)d\sigma = 0 \quad \text{for} \quad i = 1,\ldots,m. \tag{14}$$

II - Discretization

The discretization of (9-10), or of (9,13) in the linear case, proceeds in the usual manner, with the only difficulty resulting from the inhomogeneity q in the essential boundary condition (5). If the boundary Γ consists of polygons, then one merely chooses a finite dimensional subspace $V^h \subset V$, and then define the space

$$V^h_a = \{\phi\ \varepsilon\ V^h;\ \phi = 0 \text{ on } \Gamma_o;\ \phi = c^h_i \text{ on } \Gamma_i,$$
$$i = 1,\ldots,m,\ c^h_i \text{ arbitrary}\}$$

and the set

$$S^h_a = \{\phi\ \varepsilon\ V^h;\ \phi = q^h \text{ on } \Gamma_o;\ \phi = q^h+c^h_i \text{ on } \Gamma_i,$$
$$i = 1,\ldots,m,\ c^h_i \text{ arbitrary}\}$$

where $q^h\ \varepsilon\ V^h$ is an approximation to q. For example, since one need define q^h only along boundaries, we may choose q^h on Γ to be the boundary interpolant of q in V^h restricted to Γ. Then one seeks $\psi^h\ \varepsilon\ S^h_a$ and $\omega^h\ \varepsilon\ V^h$ such that

$$\int_\Omega \omega^h\zeta^h\ d\Omega - \int_\Omega \underline{\text{curl}}\ \psi^h\cdot\underline{\text{curl}}\ \zeta^h\ d\Omega \tag{15}$$

$$= \int_\Gamma \zeta^h\underline{g}\cdot\underline{\tau}\ d\sigma \quad \text{for all} \quad \zeta^h\ \varepsilon\ V^h,$$

$$-\nu\int_\Omega \underline{\text{curl}}\ \omega^h\cdot\underline{\text{curl}}\ \phi^h\ d\Omega + \int_\Omega \omega^h\left(\frac{\partial\psi^h}{\partial y}\frac{\partial\phi^h}{\partial x}-\frac{\partial\psi^h}{\partial x}\frac{\partial\phi^h}{\partial y}\right)d\Omega \tag{16}$$

$$= -\int_\Omega \underline{f}\cdot\underline{\text{curl}}\ \phi^h\ d\Omega \quad \text{for all} \quad \phi^h\ \varepsilon\ V^h_a.$$

Curved boundaries may be treated by isoparametric finite elements and similar techniques. The linear Stokes equations can be discretized in the same manner with $V^h \subset H^1(\Omega)$.

The discrete weak formulation (15-16) is equivalent to a system of nonlinear algebraic equations. The latter is arrived at by choosing a basis $\{\phi_j\}$, $j = 1,\ldots,J$, for V^h. Suppose we order these basis functions so that

$$\phi_j = 0 \text{ on } \Gamma \text{ for } j = 1,\ldots,N$$

$$\phi_j = 0 \text{ on } \Gamma - \Gamma_i \text{ for } j = N+M_{i-1}+1,\ldots,N+M_i \text{ and for } i = 0,\ldots,m, \tag{17}$$

where $M_{-1} = 0$ and $N+M_m = J$. Thus the first N basis functions correspond to interior degrees of freedom or to degrees of freedom on Γ which are not associated with function values and the basis functions indexed by $j = N+M_{i-1}+1,\ldots,N+M_i$ correspond to the degrees of freedom associated with function values on Γ_i. For example, if V^h is a Lagrangian finite element space, then ϕ_j for $j = 1,\ldots,N$ correspond to interior nodes while ϕ_j for $j = N+M_{i-1}+1,\ldots,N+M_i$ correspond to nodes on Γ_i. Thus, in this example, N is the number of nodes in the interior of Ω and $(M_i - M_{i-1})$ are the number of nodes on Γ_i for $i = 0,\ldots,m$. We note that in a practical implementation one would not choose to number the basis functions according to (17). We merely choose that numbering scheme in order to simplify the exposition which follows.

Having chosen the numbering scheme (17), we may then substitute in (15-16)

$$\omega^h = \sum_{j=1}^{J} \omega_j \phi_j(\underline{x})$$

and

$$\psi^h = \sum_{j=1}^{N} \psi_j \phi_j(\underline{x}) + \sum_{j=N+1}^{J} q(\underline{x}_j)\phi_j(\underline{x}) + \sum_{i=1}^{m} a_i^h \sum_{j=N+M_{i-1}+1}^{N+M_i} \phi_j(x) \tag{18}$$

where $\underline{x}_j$ denotes the coordinates of the j-th node and where we have approximated q on Γ by its boundary interpolant. We may also choose, in (15-16), $\zeta^h = \phi_k(\underline{x})$ for $k = 1,\ldots,J$ and $\phi^h = \phi_k(x)$ for $k = 1,\ldots,N$ and $\phi^h = \sum \phi_j(x)$ where the sum ranges over $j = N+M_{i-1}+1,\ldots,N+M_i$ for $i = 1,\ldots,m$. Thus (15-16) represent $(N+J+m)$ nonlinear algebraic equations for the $(N+J+m)$ unknowns ω_j, $j = 1,\ldots,J$, ψ_j, $j = 1,\ldots,N$, and a_i^h, $i = 1,\ldots,m$. Below we remark on how to solve this discrete system.

For the case of the linear Stokes problem, the weak formulation (15-16), where now the nonlinear terms in (16) are not present, leads, in an analogous manner to a linear system of $(N+J+m)$ algebraic equations.

Nonlinear Solver

The nonlinear system of equations resulting from (15-16) may be solved in a variety of ways. We are particularly interested in preserving the feature that no boundary conditions on the vorticity be imposed at boundaries for which the velocity is known. This precludes the use of algorithms which iterate between (15) and (16). Even so, one may choose to imbed (15-16) into a real or pseudo-time dependent problem and then discretize the time derivatives, or one may choose to use a nonlinear equation solver such as Newton's method, a quasi-Newton method, or even a fixed point method. Because of its robustness and ease of programming, we use Newton's method, although quasi-Newton methods, when applicable, may be more efficient. We here give the Newton algorithm which is most easily described in terms of the weak formulation. To further simplify the presentation we introduce the bilinear forms

$$A(\omega,\zeta) = \int_\Omega \omega\zeta \, d\Omega \quad \text{for} \quad \omega,\zeta \in V \tag{19}$$

and

$$B(\psi,\zeta) = -\int_\Omega \underline{\text{curl}}\ \psi \cdot \underline{\text{curl}}\ \zeta \, d\Omega \quad \text{for} \quad \psi,\zeta \in V, \tag{20}$$

the trilinear form

$$C(\omega,\psi,\phi) = \int_\Omega \omega\left(\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y}\right) d\Omega \text{ for } \omega,\psi,\zeta \in V, \tag{21}$$

and the linear functionals

$$F(\phi) = -\int_\Omega \underline{f}\cdot\underline{\text{curl}}\ \phi \, d\Omega \quad \text{and} \quad G(\phi) = \int_\Gamma \phi\underline{g}\cdot\underline{\tau} \, d\Omega \text{ for } \phi \in V. \tag{22}$$

Then, (15-16) take the form

$$A(\omega^h,\zeta^h) + B(\psi^h,\zeta^h) = G(\zeta^h) \text{ for all } \zeta^h \in V^h, \tag{23}$$

$$\nu B(\phi^h,\omega^h) + C(\omega^h,\psi^h,\phi^h) = F(\phi^h) \text{ for all } \phi^h \in V_a^h. \tag{24}$$

Newton's method for (23-24) is then given as follows. Given $\omega^o \in V^h$ and $\psi^o \in S_a^h$, we generate the sequence $\{\omega^n,\psi^n\}$ for $n \geq 1$ by solving the linear system

$$A(\omega^n,\zeta^h) + B(\psi^n,\zeta^h) = G(\zeta^h) \quad \text{for all} \quad \zeta^h \in V^h \tag{25}$$

$$\nu B(\phi^h,\omega^n) + C(\omega^n,\psi^{n-1},\phi^h) + C(\omega^{n-1},\psi^n,\phi^h) = F(\phi^h) + C(\omega^{n-1},\psi^{n-1},\phi^h) \quad \text{for all} \quad \phi^h \in V_a^h. \tag{26}$$

Due to the relatively small attraction ball of Newton's method, it is convenient to start with the following simple iteration method which, at least at low Reynolds numbers, is globally convergent. Given $\omega^o \in V^h$ and $\psi^o \in V_a^h$, we generate the sequence $\{\omega^n,\psi^n\}$ for $n \geq 1$ by solving the linear system

$$A(\omega^n,\zeta^h) + B(\psi^n,\zeta^h) = G(\zeta^h) \text{ for all } \zeta^h \in V^h \tag{27}$$

and

$$\nu B(\phi^h,\omega^n) + C(\omega^{n-1},\psi^n,\phi^h) = F(\phi^h) \text{ for all } \phi^h \in V_a^h. \tag{28}$$

Thus the composite algorithm consists of doing a few steps, usually one or two, of (27-28) in order to get into the neighborhood of the solution of (23-24), and then switching to Newton's method (25-26) in order to more quickly home in on that solution. Note that the simple iteration algorithm (27-28), and therefore the composite algorithm as well, do not need the initial guess ψ^o to satisfy the boundary conditions. In particular, we may choose $\psi^o = \omega^o = 0$.

By methods similar to those used in [1,5,6] for the primitive variable, i.e., velocity-pressure, formulation it can be shown that the Newton iterates defined by (25-26) converge quadratically to a solution of (23-24) for a sufficiently close initial guess. Further remarks concerning the solution of the discrete nonlinear system (23-24) are made below when computational results are discussed. We note that either of the methods (25-26) or (27-28) require the solution of a sequence of linear algebraic systems.

Two Methods for Multiply Connected Domains

As was previously discussed, the main difficulty which multiply connected domains present is that the streamfunction may be arbitrarily specified on only one part of the boundary. Thus, above, we have set ψ and $\psi^h = 0$ at one point on Γ_o, but these are determined only up to the unknown constants a_i and a_i^h, respectively, on the remaining parts Γ_i, $i = 1,\ldots,m$, of the boundary. In the discretization method represented by (15-16), the constants a_i^h are determined as part of the solution process. However, that method requires the use of

$$\sum_{j=N+M_{i-1}+1}^{N+M_i} \phi_j(x), \quad i = 1,\ldots,m \tag{29}$$

as basis functions associated with the a_i^h, $i = 1,\ldots,m$. In general, these basis functions are only "semi-local" in the sense that they couple all the points on a boundary Γ_i. In particular, in the discrete set of equations, a_i^h is coupled with all the unknowns associated with elements whose closures intersect with Γ_i. The result of this is that the bandwidth of the linear systems encountered will be greatly increased, resulting in both a computer storage and time penalty over simply connected domain problems. Some of this penalty may be mitigated by employing a numbering scheme resulting in banded-bordered matrices wherein the onerous couplings are found near the bottom and the right of the matrices encountered. This, of course, requires the development of special programs to solve the linear systems.

An alternative to using basis functions such as (29) is to guess the value of the constants a_i, $i = 1,\ldots,m$, appearing in (5). Then one may solve for ψ and ζ from (1,2,5 and 6). However, in general, (8) will not be satisfied. At this point one may change the guesses for the a_i's, repeating the process until convergence is achieved, i.e., until (8) is satisfied. In more detail, we proceed as follows. Given guesses a_i^k, $i = 1,\ldots,m$, $k \in Z^+$, we define $\psi_k^h \in S_k$ and $\omega_k^h \in V^h$ where

$$S_k = \{\phi \in H^1(\Omega) \cap W^{1,4}(\Omega);\ \phi = q \text{ on } \Gamma_o;$$
$$\phi = q + a_i^k \text{ on } \Gamma_i,\ \ i = 1,\ldots,m\},$$

to be the solution of (15-16) where now (16) holds for all $\xi^h \in V_o^h = V^h \cap H_o^1(\Omega)$. In this case ψ_k^h is given by (18) with a_i^h replaced by the known numbers a_i^k and thus the discrete system (15-16) contains (N+J) equations for the (N+J) unknowns ω_j, $j = 1,\ldots,J$, and ψ_j, $j = 1,\ldots,N$. Furthermore, these equations and unknowns may be ordered in such a way that the bandwidth of the discrete system is no larger than that obtainable in an analogous discretization of a problem posed in the simply connected domain bounded by Γ_o. Having obtained ω_k^h from a_i^k, $i = 1,\ldots,m$, we may substitute into (8). In practice, we would evaluate the integrals appearing in (8) by a numerical quadrature formula. Let us denote such an approximation to each equation in (8) by $I_i(\underline{a}^k)$, $i = 1,\ldots,m$, which in general will not be small. Of course, each $I_i(\cdot)$ is a nonlinear function of $\underline{a}^k = (a_1^k,\ldots,a_m^k)$ through the discrete vorticity ω_k^h. We now need an algorithm to generate new guesses $\underline{a}^{k+1}$, from which ψ_{k+1}^h and ω_{k+1}^h may be computed and for which $I_i(\underline{a}^{k+1})$ is closer to zero than was $I_i(\underline{a}^k)$. The most practical way to accomplish this is to use a secant type method. For example, if $m = 1$, i.e., Ω is doubly connected, we may then define

$$a_1^{k+1} = a_1^k - \frac{a_1^k - a_1^{k-1}}{I_1(a_1^k) - I_1(a_1^{k-1})} I_1(a_1^k). \tag{30}$$

For $m > 1$, some generalized secant method in R^m such as the Wolfe secant method can be used. See [7] for details about such methods. Note that the use of formulas such as (30) requires two starting guesses $\underline{a}^o$ and $\underline{a}^1$. We also note that in general it is not practical to use Newton's method to update $\underline{a}^k$ since evaluating the Jacobian of $\underline{I}(a) = (I_1,\ldots,I_m)$ requires the solution of linear problems of the same size as (15-16) for each pair $\partial\omega/\partial a_i$, $\partial\psi/\partial a_i$.

Thus, at the price of solving a sequence of simpler problems, one may avoid the bandwidth problems engendered by the use of basis functions such as (29). Whether or not the alternate method is more efficient depends, of course, on how many of the simpler problems must be solved as well as on how much cheaper it is to solve the simpler problems. Both of these factors, for a particular geometry, will depend on the number of degrees of freedom, i.e., the gridsize, and the Reynolds number. Therefore it is clear that the performance of iterations such as (30) play a central role in the overall efficiency of the alternate method of treating multiply connected domains. Unfortunately, the sequence $\underline{a}^k$ generated by secant type methods are not guaranteed to converge for arbitrary values of the initial guesses, especially at high values of the Reynolds number. We will return to this point when we discuss some computational results below.

Multiply Connected Domains in the Linear Case

The situation for the alternate method discussed above is much simpler in the case of the linear Stokes equations. In this case, $I(\underline{a})$ is a linear function of the a_i's and thus, given any $\underline{a}^o$ and $\underline{a}^1$, a secant method such as (30) will yield that $\underline{a}^2$ is the exact desired value $\underline{a}^k$. Furthermore, ω_2^h and ψ_2^h are linear combinations of ω_k^h and ψ_k^h, $k = 0,1$, and thus the former may be obtained from the latter without solving another linear problem of the type (15-16) where in (16) the nonlinear terms are omitted.

These observations may be used to define an even simpler algorithm for the linear case which is equivalent to the above alternate method. First we solve (m+1) discrete problems corresponding to the continuous problems

$$\Delta\psi_k = -\omega_k \quad \text{and} \quad -\nu\Delta\omega_k = \text{curl } \underline{f}^k \quad \text{in } \Omega,$$

$$\psi_k = q^k \text{ on } \Gamma_o, \quad \psi_k = q^k + c_i^k \text{ on } \Gamma_i, \quad i = 1,\ldots,m \tag{31}$$

and

$$\frac{\partial\psi_k}{\partial n} = -\underline{g}^k\cdot\underline{\tau} \quad \text{on } \Gamma$$

where $k = 0,\ldots,m$ and $\underline{f}^o = \underline{f}$, $q^o = q$, $\underline{g}^o = \underline{g}$, $c_i^o = 0$, $\underline{f}^k = q^k = \underline{g}^k = 0$, and $c_i^k = \delta_{ik}$ for $k \geq 1$ and $i = 1,\ldots,m$. All of these problems involve the same left hand side, i.e., the same coefficient matrix in the discrete problem, and thus may be solved simultaneously by setting up $(m+1)$ right hand sides. Now consider the combination

$$\bar\psi = \psi_o + \sum_{k=1}^m a_k\psi_k \quad \text{and} \quad \bar\omega = \omega_o + \sum_{k=1}^m a_k\psi_k \tag{32}$$

where a_k, $k = 1,\ldots,m$, are constants to be determined. Then, since the problems (31) are linear, we have that

$$\Delta\bar\psi = -\bar\omega \quad \text{and} \quad -\nu\Delta\bar\omega = \text{curl } \underline{f} \quad \text{in } \Omega$$

$$\bar\psi = q \text{ on } \Gamma_o, \quad \bar\psi = q + a_i \text{ on } \Gamma_i, \quad i = 1,\ldots,m,$$

and

$$\frac{\partial\bar\psi}{\partial n} = -\underline{g}\cdot\underline{\tau} \quad \text{on } \Gamma.$$

Thus, for any values of the constants a_i, $\bar\psi$ and $\bar\omega$ satisfy the given Stokes problem (1,5,6 and 13). We fix the a_i's by requiring that (14) be satisfied. Indeed, denoting each of the integrals in (14) by $I_i(\omega)$, we see that since these are linear in ω

$$0 = I_i(\bar\omega) = I_i(\omega_o) + \sum_{k=1}^m a_k I_i(\omega_k) \quad \text{for } i = 1,\ldots,m. \tag{33}$$

Thus (33) is a linear system of m equations for the m constants a_k, $k = 1,\ldots,m$. Having found these constants, then (32) yields the solution of the given Stokes problem.

Of course, the use of the continuous problems (31) was symbolic; in reality one solves the corresponding discrete problems and approximates the integrals $I_i(\omega_k)$ by numerical quadrature. In summary, we see that in the linear case, we may solve multiply connected domain problems by solving a single "simple" matrix problem, with multiple right hand sides, evaluating the $(m+1)m$ integrals $I_i(\omega_k)$, $k = 0,\ldots,m$, $i = 1,\ldots,m$, solving the $m \times m$ linear system (33), and then taking the linear combinations (32). On the other hand, one may solve the same problem by using basis functions such as (29). In this case one need solve a single linear system with a single right hand side to obtain the solution. However, as noted above, this system will surely be more complicated than that for the method (31-33), and it seems that, at least for moderate values of m, the latter technique is preferable.

Error Estimates

We now turn to a brief discussion of the available theoretical estimates of the differences $\psi-\psi^h$ and $\omega-\omega^h$. We will consider the case of $q = 0$ and $g = 0$ and of simply connected domains. Inhomogeneous problems can be treated by techniques similar to those used in [8] for the primitive variable formulation of Navier-Stokes equations and insofar as the accuracy of the approximate solution, problems posed on multiply connected domains should behave in a manner similar to those posed on simply connected domains. Furthermore, we will focus mostly on low order, low continuity finite element spaces, especially continuous piecewise linear polynomial spaces.

Most of the results available are concerned with the linear Stokes problems in a simply connected domain with homogeneous boundary data. Indeed, finite element discretizations of the particular weak form (9,13) are considered in [9-14]. All except for [13] and [14] consider only the case of piecewise polynomial spaces of degree two or higher. The analyses of [14] improves on the previous works, and it is the results of that work which we summarize here.

To begin with, we define the space of weakly harmonic functions

$$H = \{\zeta \in H^1(\Omega);\ \int_\Omega \nabla\zeta\cdot\nabla\phi \, d\Omega = 0 \quad \text{for all} \quad \phi \in H_o^1(\Omega)\}.$$

We also define the norms

$$\|\phi\|_{-1} = \sup_{\zeta\in H^1(\Omega)} \frac{\int_\Omega \zeta\phi \, d\Omega}{\|\zeta\|_1} \quad \text{and} \quad \|\phi\|_* = \sup_{\zeta\in H} \frac{\int_\Omega \zeta\phi \, d\Omega}{\|\zeta\|_1} \tag{34}$$

where $\|\cdot\|_r$ for $r \in Z^+$ denotes a norm on $H^r(\Omega)$. Note that the first of these is not the norm on $H^{-1}(\Omega)$, the dual space of $H_o^1(\Omega)$, but we adopt this notation for simplicity. A basic result is that the two norms of (34) are equivalent on H. (The proof of this and the other statements concerning the linear case may be found in [14].)

Now, recall that our approximations are based on choosing a subspace $V^h \subset H^1(\Omega)$ which we will assume to be a continuous piecewise polynomial space, and then letting $V_o^h = V^h \cap H_o^1(\Omega)$. We may then define the space of discrete weakly harmonic functions

$$H^h = \{\zeta^h \in V^h;\ \int_\Omega \underline{\text{curl}}\,\zeta^h\cdot\underline{\text{curl}}\,\phi^h \, d\Omega = 0 \text{ for all } \phi^h \in V_o^h\}.$$

In general $H^h \not\subset H$. In fact, the lack of this inclusion results in the non-optimality of the approximations to ω and also causes the main difficulty in the analyses. Having defined H^h, we may also define the norm

$$\|\phi\|_{*,h} = \sup_{\zeta^h\in H^h} \frac{\int_\Omega \zeta^h\phi^h \, d\Omega}{\|\zeta^h\|_1}. \tag{35}$$

In general the norms $\|\cdot\|_{*,h}$ and $\|\cdot\|_{-1}$ are not equivalent on H^h. However, for any $\phi^h \in H^h$ and $\varepsilon > 0$ we have that for some constants C_1 and C_2,

$$\|\phi^h\|_{*,h} \leq \|\phi^h\|_{-1} \leq C_1\|\phi^h\|_{*,h} + C_2 h^{1/2-\varepsilon}\|\phi^h\|_o. \tag{36}$$

We next define the differences

$$e_1^h = \psi^h - \hat{\psi}^h,\ e_1 = \psi - \hat{\psi}^h,\ e_2^h = \omega^h - R_h\omega \text{ and } e_2 = \omega - R_h\omega \quad (37)$$

where $\hat{\psi}^h \in V_o^h$ is arbitrary and $R_h\omega \in V^h$ satisfies

$$\int_\Omega \underline{\text{curl}}(\omega - R_h\omega)\cdot\underline{\text{curl}}\ \phi^h\, d\Omega = 0 \quad \text{for all} \quad \phi^h \in V_o^h. \quad (38)$$

Also, we note that from (9,13) with $\underline{g} = 0$ and $q = 0$ and from its discrete analogue,

$$\int_\Omega \underline{\text{curl}}\ e_2^h\cdot\underline{\text{curl}}\ \phi^h\, d\Omega = 0 \quad \text{for all} \quad \phi^h \in V_o^h \quad (39)$$

$$\int_\Omega (e_2^h\zeta^h + \underline{\text{curl}}\ e_1^h\cdot\underline{\text{curl}}\ \zeta^h)d\Omega = \int_\Omega (e_2\zeta^h + \underline{\text{curl}}\ e_1\cdot\underline{\text{curl}}\ \zeta^h)d\Omega$$
$$\text{for all} \quad \zeta^h \in V^h. \quad (40)$$

We then obtain that for any $\varepsilon > 0$ and for some constants C_3, C_4 and C_5,

$$\|\omega-\omega^h\|_o \le 2\|\omega - R_h\omega\|_o + E_h^o \quad (41)$$

and

$$\|\underline{\text{curl}}(\psi-\psi^h)\|_o \le 2 \inf_{\hat{\psi}^h\in V_o^h} \|\underline{\text{curl}}(\psi-\hat{\psi}^h)\|_o \quad (42)$$
$$+ C_3\|\omega-R_h\omega\|_o + C_4 E_h^1 + C_5 h^{1/2-\varepsilon}E_h^o$$

where

$$E_h^j = \inf_{\hat{\psi}^h\in V_o^h} \sup_{\zeta^h\in H^h} \frac{\int_\Omega \underline{\text{curl}}(\psi-\hat{\psi}^h)\ \underline{\text{curl}}\ \zeta^h\, d\Omega}{\|\zeta^h\|_j} \quad (43)$$

for $j = 0,1$.

To prove (41) and (42), first note that by (39), $e_2^h \in H^h$ and from (40) with $\zeta^h \in H^h$ one easily obtains that

$$\|e_2^h\|_{*,h} \le \|e_2\|_{-1} + \sup_{\zeta^h\in H^h} \frac{\int_\Omega \underline{\text{curl}}\ e_1\cdot\underline{\text{curl}}\ \zeta^h\, d\Omega}{\|\zeta^h\|_1} \quad (44)$$

$$\|e_2^h\|_o \le \|e_2\|_o + \sup_{\zeta^h\in H^h} \frac{\int_\Omega \underline{\text{curl}}\ e_1\cdot\underline{\text{curl}}\ \zeta^h\, d\Omega}{\|\zeta^h\|_o}. \quad (45)$$

Also (36) and (40) yield that

$$\|\underline{\text{curl}}\ e_1^h\|_o \le \|e_2^h\|_{-1} + \|e_2\|_{-1} + \|\underline{\text{curl}}\ e_1\|_o \quad (46)$$
$$\le C_1\|e_2^h\|_{*,h} + C_2 h^{1/2-\varepsilon}\|e_2^h\|_o + \|e_2\|_{-1}$$
$$+ \|\underline{\text{curl}}\ e_1\|_o.$$

Then (42) follows from (37), (45), and the triangle inequality, and (41) follows from (37), (44-46), and the triangle inequality.

If $H^h \subset H$, then $E_h^o = E_h^1 = 0$. Thus (41) and (42) yield that

$$\|\omega-\omega^h\|_o \le 2\|\omega-R_h\omega\|_o$$

$$\|\underline{\text{curl}}(\psi-\psi^h)\|_o \le 2 \inf_{\hat{\psi}^h\in V_o^h} \|\underline{\text{curl}}(\psi-\hat{\psi}^h)\|_o + C_3\|\omega-R_h\omega\|_o$$

which are optimal estimates for the L^2-norm of the vorticity and of the velocity field. Unfortunately, the inclusion $H^h \subset H$ implies that $V^h \subset H^2(\Omega)$, i.e., V^h consists of continuously differentiable piecewise polynomials. In this case we might as well have discretized, using $V^h \subset H^2(\Omega)$, the fourth order streamfunction formulation of the Stokes equations.

Of more interest to us here is the case of $V^h \not\subset H^2(\Omega)$, e.g., V^h consisting of merely continuous piecewise polynomials. Examining (41) and (42) yields that except for the terms involving E_h^o and E_h^1, the terms on the right hand sides are either approximation theoretic, or, due to (38), can be approximated in terms of approximation theoretic results [14]. Thus it only remains to estimate, for specific choices of V^h, the terms E_h^o and E_h^1. For example, if V^h consists of piecewise linear polynomials and if $\psi \in H^2(\Omega) \cap H_o^1(\Omega)$, then for any $\varepsilon > 0$ and $j = 0,1$, it can be shown that

$$E_h^j \le C_{6+j}\, h^{j+\frac{1}{2}-\varepsilon}\|\psi\|_2.$$

Then (41) and (42) yield, for linear finite element spaces,

$$\|\omega-\omega^h\|_o = O(h^{1/2}) \quad \text{and} \quad \|\underline{\text{curl}}(\psi-\psi^h)\|_o = O(h^{1-\varepsilon}). \quad (47)$$

We note that (47) shows that the velocity field $\underline{u} = \underline{\text{curl}}\ \psi$ is optimally approximated and that there is a 3/2's loss in the exponent of h in the estimate for ω.

For the nonlinear case, there are much fewer error estimates available in the literature. In fact, for the particular method considered here, such estimates can only be found in [1]. Furthermore, these hold only for polynomial spaces of degree two or higher, and yield suboptimal results. However, by combining the nonlinear analysis of [1] with the improved analysis, for the linear problem, found in [14], results such as those above can be reasonably expected to hold for the nonlinear problem as well.

III - Remarks

Infinite Domain Problems - So far our considerations have focused on problems which may be posed on bounded domains. However, the streamfunction-vorticity formulation may be especially useful for problems posed on exterior domains since for many such problems the vorticity decays exponentially with the distance from the origin while the velocity and pressure only decay algebraically. Thus applying far field conditions on the vorticity can lead to substantially smaller computational regions than that needed for a computation of the same accuracy using such conditions on the velocity. We have also indicated in Section I how easily implemented boundary conditions on the vorticity may be useful in matching to external inviscid flows in boundary layers and wake type calculations.

Finite Difference Methods - It would be remiss not to point out that finite difference methods may also be implemented in such a manner that no artificial boundary condition on the vorticity is imposed at solid boundaries. Indeed, the key to avoiding such a specification is not the use of finite element methods. Rather it is the willingness to solve the discrete system (15-16) as one coupled set of equations as opposed to iterating between (15) and (16). Thus, in a finite difference method, one could discretize (1) and (6) while making use of (5) as well, yielding more equations than unknowns determining ψ as a function of ω and the boundary data. One also discretizes (2) without imposing any boundary condition on ω, yielding less equations than unknowns to "determine" ω. In this manner one may not solve the discrete versions of (1,5,6) for ψ since that system is overconstrained, and also one may not uniquely solve the discrete version of (2) for ω since that system is underconstrained. However, the above type of discretizations of (1,2,5 and 6), taken together, may be solved simultaneously for ψ and ω. We note that similar observations about the relation of the discrete systems resulting from (1,5,6) and (2) hold in the finite element case.

The method of treating multiply connected domains wherein one uses iterations such as (30) can clearly be used in connection with finite difference methods where again one must approximately evaluate the constraint (8) or (14) in order to update the guessed values of ψ at the boundaries. Also in a straightforward manner, one may implement the other method. Specifically, one can leave the constants a_i in (5) as unknowns, and then add a discretization of (8) or (14) in order to close the system of equations. However, we note that in the finite element case (8) or (14), as well as (6), are natural boundary conditions, and thus are more easily satisfied. The same remarks concerning the relative efficiency, e.g., bandwidth size, of the two methods which held for the finite element case also hold in the finite difference case.

Recovery of the Primitive Variables - The computation of the velocity field from the streamfunction is a simple matter since we may define $\underline{u}^h = (\partial\psi^h/\partial y, -\partial\psi^h/\partial x)$. The recovery of the pressure field is not so straightforward, especially for low continuity, e.g., merely continuous, finite element spaces. Formally, one would like to use (7) in the following way. From (7), we define a discrete pressure $p^h \in V^h \subset H^1(\Omega) \cap W^{1,4}(\Omega)$ such that

$$\int_\Omega \nabla p^h \cdot \nabla \hat{p}^h \, d\Omega = \rho \int_\Omega (-\underline{u}^h \cdot \nabla \underline{u}^h + \nu \nabla \underline{u}^h + \underline{f}) \cdot \nabla \hat{p}^h \, d\Omega \qquad (48)$$

$$\text{for all } \hat{p}^h \in V^h$$

where $\underline{u}^h$ in the right hand side is obtained from the already computed approximate streamfunction ψ^h. Note that, by integrating by parts, that (48) may be viewed formally as a discretization of the problem

$$\Delta p = \rho \operatorname{div}(-\underline{u} \cdot \nabla \underline{u} + \nu \Delta \underline{u} + \underline{f}) \quad \text{in } \Omega$$

$$\frac{\partial p}{\partial n} = \rho(-\underline{u} \cdot \nabla \underline{u} + \nu \Delta \underline{u} + \underline{f}) \cdot \underline{n} \quad \text{on } \Gamma.$$

Both of these may be obtained directly from (7) by respectively taking the divergence of (7) and the inner product of (7) with $\underline{u}$. Unfortunately, the right hand side of (48) is not defined when one uses merely continuous finite element spaces for ψ^h and ω^h. The problem is not with the viscous terms since, from the definition of $\underline{u}^h$, we have that

$$\int_\Omega \Delta \underline{u}^h \cdot \nabla \hat{p}^h \, d\Omega = \int_\Gamma \hat{p} \frac{\partial}{\partial \tau} (\Delta \psi^h) d\sigma = -\int_\Gamma \hat{p} \frac{\partial \omega^h}{\partial \tau} d\sigma \qquad (49)$$

where the last equality is only approximate. The integral on the right is well defined, even for low continuity spaces. The problematical term is the convection term in (48) which is not defined for merely continuous spaces. However, for the linear case or the nonlinear case with finite element spaces which are at least continuously differentiable, one may use (48), with the replacement (49), to solve for the pressure.

We now present a method for recovering the pressure which works in all cases. First, (7) may be expressed in the form

$$\nabla H = \rho[\binom{v}{-u}\omega + \underline{f} + \nu \Delta \underline{u}] \text{ where } H = (p + \tfrac{1}{2} \rho \underline{u} \cdot \underline{u}), \qquad (50)$$

i.e., H is the total pressure head. We then consider the problem of seeking $H \in H^1(\Omega) \cap W^{1,4}(\Omega)$ such that

$$\int_\Omega \nabla H \cdot \nabla \hat{H} \, d\Omega = \rho \int_\Omega (-\omega \underline{u} \cdot \underline{\operatorname{curl}} \, \hat{H} + \underline{f} \cdot \nabla \hat{H} + \nu \nabla \hat{H} \cdot \Delta \underline{u}) d\Omega$$

$$= \rho \int_\Omega (-\omega \underline{u} \cdot \underline{\operatorname{curl}} \, \hat{H} + \underline{f} \cdot \nabla \hat{H}) d\Omega - \rho\nu \int_\Gamma \hat{H} \frac{\partial \omega}{\partial \tau} d\sigma$$

$$\text{for all } \hat{H} \in H^1(\Omega) \cap W^{4.1}(\Omega)$$

which by additional integration by parts, is formally equivalent to

$$\Delta H = \rho \operatorname{curl}(\omega \underline{u}) + \rho \operatorname{div} \underline{f} \quad \text{in } \Omega$$

$$\frac{\partial H}{\partial n} = \rho(\underline{f} \cdot \underline{n} - \nu \frac{\partial \omega}{\partial \tau} + \omega \underline{u} \cdot \underline{\tau}) \quad \text{on } \Gamma. \qquad (51)$$

These, using $\operatorname{div} \underline{u} = 0$, may be derived directly from (50). Thus we are led to define an approximation $H^h \in V^h$, the same finite element space used for the vorticity approximation, by the solution of the following problem. We seek $H^h \in V^h$ such that

$$\int_{\Omega} \nabla H^h \cdot \nabla \hat{H}^h \, d\Omega = \rho \int_{\Omega} (\underline{f} - \omega^h \nabla \psi^h) \cdot \nabla \hat{H}^h \, d\Omega \tag{52}$$

$$- \rho\nu \int_{\Gamma} \hat{H}^h \frac{\partial \omega^h}{\partial \tau} \, d\sigma \text{ for all } \hat{H}^h \in V^h$$

where now the right hand side is well defined for the already computed values of $\omega^h \in V^h$ and $\psi^h \in S_a^h$. Once the approximate total head H^h is computed from (52), the pressure is easily recovered from $p^h = H^h - [(\psi_x^h)^2 + (\psi_y^h)^2]/2$. We note that the solution H^h of (52) corresponds to an approximate solution of the Neumann problem (51). Thus H^h and therefore p^h are determined only up to an additive constant, which is to be expected.

Three Dimensional Problems - The streamfunction-vorticity formulation of the Navier-Stokes equations has not achieved the same interest or success in three dimensional settings as it has in two dimensions. However, recently there has been increasing attention devoted to such problems. See, e.g., [15,16] for finite difference approximations of three dimensional problems.

Since $\operatorname{div} \underline{u} = 0$, we have that necessarily $\underline{u} = \operatorname{curl} \underline{\psi}$ for some vector valued function $\underline{\psi}$ variously called the "vector streamfunction" or the "vector velocity potential". Of course, the vorticity is defined to be $\underline{\omega} = \operatorname{curl} \underline{u}$. Then, the Navier-Stokes equations, in terms of the streamfunction $\underline{\psi}$ and vorticity $\underline{\omega}$, are given by

$$\operatorname{curl} \operatorname{curl} \underline{\psi} = \underline{\omega} \tag{53}$$

$$\nu \operatorname{curl} \operatorname{curl} \underline{\psi} = \operatorname{curl} \underline{\psi} \cdot \nabla \underline{\omega} - \underline{\omega} \cdot \nabla (\operatorname{curl} \underline{\psi}) \tag{54}$$

$$= \operatorname{curl}(\underline{\omega} \times \operatorname{curl} \underline{\psi}).$$

At a boundary where $\underline{u}$ is specified, one would now specify $\operatorname{curl} \underline{\psi}$.

As a consequence of their definition, it follows that $\underline{\psi}$ can be determined only up to the gradient of an arbitrary scalar function, and that $\operatorname{div} \underline{\omega} = 0$. There are various ways in which these facts have been used to simplify the formulation. For example, the arbitrariness in $\underline{\psi}$ can be pinned down by requiring that $\operatorname{div} \underline{\psi} = 0$. This method, popular in electromagnetic problems where it is called the "Coulomb gauge", has been used in [15] in connection with a finite difference solution of vortex flows in all of $\mathbb{R}^3$, i.e., a problem with no boundaries. In problems with solid boundaries, it becomes difficult to enforce boundary conditions. The obvious disadvantage of this method is that there are six unknown scalar fields. The main advantage of this method is that (53) and $\operatorname{div} \underline{\psi} = 0$ imply that $-\Delta\underline{\psi} = \underline{\omega}$, so that together with (54), the governing equations may be viewed as coupled Poisson equations for the six scalar fields constituting the components of $\underline{\psi}$ and $\underline{\omega}$. Although more fields are required than in the primitive variable formulation and as many as in the velocity-vorticity formulation, the resulting streamfunction vorticity equations are presumably easier to solve. Furthermore, the finite element techniques discussed in this paper for the plane flow setting extend in a straightforward manner to this three dimensional method.

Another way of fixing the streamfunction is to set one component, say the component ψ_1 in the x-direction, to zero. The obvious advantage of this method, which was used successfully in [16] to compute compressible flows, is that we have only two unknown streamfunction fields. In addition, as is indicated in [16], this method can handle solid boundaries better than the first technique and may be especially useful for flows which are perturbations of two dimensional or axially symmetric flows. In [16] a third technique is also discussed, namely letting $\underline{\psi} = \nabla\phi \times \nabla\sigma$ which also involves only two scalar fields. However, this choice introduces additional nonlinearities into the problem which, in practice, is not desirable.

IV - Computational Examples

Accuracy - We first examine, through computational examples, the accuracy of the particular finite element methods discussed in Section II. The context of these examples is smooth solutions of the linear Stokes equations posed on the unit square, i.e., a simply connected domain. However, we note that the algorithms have been successfully used to compute solutions of the nonlinear Navier-Stokes equations. For example, in [4] these methods are used, in conjunction with a reduced basis/continuation technique, to compute accurate approximations of driven cavity flows at Reynolds numbers up to 10,000 using relatively coarse nonuniform meshes.

Some of the computational results for the Stokes equations are summarized in Tables 1 and 2 which respectively deal with piecewise linear and piecewise quadratic finite element spaces for both ψ and ω. Each table gives the exact solutions for ψ and ω. In some of the cases of Table 2 this required the introduction of an inhomogeneity in (1), i.e., we have that $\Delta\psi + \omega = \sigma$ for some function σ. Other information contained in the tables is whether or not the boundary conditions on ψ and $\partial\psi/\partial n$ are homogeneous, i.e., whether or not q and g in (5) and (6) vanish, and also whether a uniform grid or a graded nonuniform grid was used in the calculations. Finally, each table contains the computed rates of convergence of the finite element approximations to ψ and ω as measured in the $L^2(\Omega)$ and $H^1(\Omega)$ norms, i.e., respectively the mean square errors in the function values and in the derivatives. These rates were computed by comparing errors on different grids.

The most obvious trend in these results is that the streamfunction ψ and its derivatives are always optimally approximated. On the other hand, there is in general a loss of accuracy in the vorticity approximation. In particular, there seems to be a loss of one power of h, the grid size, in the piecewise linear case, and a loss of the $3/2$ power of h in the quadratic case. Thus the theoretical results of Section II seem to be sharp in both cases for the streamfunction and also for the vorticity in the piecewise quadratic case. We also note that in the piecewise linear case we always obtain optimal approximations to ψ and ω whenever all boundary conditions are homogeneous, e.g., see 1-4 in Table 1. This was not the case for the piecewise quadratic case, e.g., see 7 in Table 2. Finally, we have also found that whenever the approximation to ψ is exact, i.e., ψ belongs to the approximating space, then the approximation to ω is optimal. See, e.g., 13 and 14 in Table 2. This result can be gleaned from (41) since in this case $E_h^o = 0$.

Multiply Connected Domains - We also report on some preliminary computations for problems posed on multiply connected domains, namely with the view of comparing the two methods of treating such problems and of examining their behavior as parameters, e.g., the grid size or Reynolds number, are varied. The domain Ω is the unit square, i.e., Γ_0 is the boundary of that square, from which we have removed a rectangle, i.e., Γ_1 is the boundary of a rectangle contained within the unit square. Thus, we deal with a doubly connected domain. In all the computations we will have that q on Γ_1 vanishes so that (5) requires that $\psi = a =$ unknown constant on that part of the boundary.

These preliminary computational results are summarized in Table 3. All computations were performed using uniform grids of size h. Results for three values of the Reynolds numbers $Re = 1/\nu$ are given and for both methods of treating multiply connected domains, i.e., using iterations such as (30) to update guessed values of the streamfunction on the boundaries or using "semi-local" basis functions such as (29) to directly compute the approximate solution. The three problems considered can be characterized by the boundary value of ψ at the left and right boundaries and the position of the hole. Specifically, we consider the following problems:

1. $\psi(0,y) = \psi(1,y) = 2y^3 - 3y^2$ $\quad \Omega_1 = (1/4,1/2) \times (1/4,3/4)$

2. same $\quad \Omega_1 = (1/4,1/2) \times (1/4,1/2)$

3. $\psi(0,y) = \psi(1,y) = y^2$ $\quad \Omega_1 = (2/5,3/5) \times (2/5,3/5)$

where Ω_1 is the region bounded by Γ_1. The remaining boundary conditions on ψ are constant values at $y = 0$ and 1 and $\partial\psi/\partial n = 0$ on all boundaries. We note that the solution of Problem 1, due to symmetries, should have $\psi = a = -1/2$ on Γ_1. We have also computed with the boundary condition of Problem 1 or 2 with the concentric hole $\Omega_1 = (1/4,3/4) \times (1/4,3/4)$, and due to the high symmetry of this configuration, the exact value of $a = -1/2$ was computed for all values of Re and h and for both methods. We also note that for the $Re = 0$ cases, i.e., for the linear Stokes equations, the results using the iterative technique always converged in the expected one step. Also, for the iterative method, the initial guesses for the secant iteration (30) were $a^0 = 1$ and $a^1 = 2/3$ in all cases.

From these preliminary results it seems that at a given value of h the direct method (B) yields better accuracy than the iterative method (A). Also computations at higher Reynolds numbers sometimes resulted in the lack of convergence of the iterative method for the above initial values. This is indicative of the possible convergence problems of the iterative method. Further and more detailed computational results will be reported on elsewhere.

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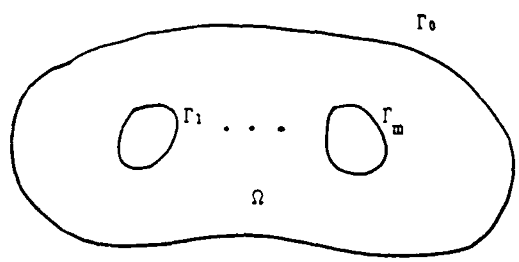

Figure 1. A multiply connected domain.

Table 1. Computational results using piecewise linear polynomials. H = homogeneous, I = inhomogeneous, U = uniform, N = nonuniform. Exact $\omega = -\Delta\psi$.

	Exact solution ψ	Boundary conditions ψ	$\partial\psi/\partial n$	Grid	Rates of convergence ψ in L^2	ω in L^2	ψ in H^1	ω in H^1
1.	$\sin^2\pi x \sin^2\pi y$	H	H	U	2	2	1	1
2.	same	H	H	N	2	2	1	1
3.	$x^2y^2(x-1)^2(y-1)^2$	H	H	N	2	2	1	1
4.	1.+ 3.	H	H	U	2	2	1	1
5.	$\cos \pi y$	I	H	U	2	1	1	0
6.	$x^2(x-1)^2$	I	H	U	2	1	1	0

Table 2. Computational results using piecewise quadratic polynomials. H = homogeneous, I = inhomogeneous. All examples use a uniform grid.

	Exact solution ψ	ω	Boundary conditions ψ	$\partial\psi/\partial n$	Rates of convergence ψ in L^2	ω in L^2	ψ in H^1	ω in H^1
7.	1.+ 3.	$-\Delta\psi$	H	H	3.3	1.7	2	.6
8.	$1 + \sin^2\pi x$	$-2\pi^2\cos 2\pi x$	I	H	3	1.5	2	.5
9.	$x^3 + y^3$	1	I	I	3	1.5	2	.5
10.	same	$x + y$	I	I	3	1.5	2	.5
11.	same	xy	I	I	3	1.5	2	.5
12.	same	x^2y	I	I	3	1.5	2	.5
13.	$x + y$	same	I	I	exact	3	exact	2
14.	$x^2 + y^2$	same	I	I	exact	3	exact	2

Table 3. Computational results for doubly connected region problems using piecewise linear functions; A = using iterative updating of ψ on boundaries, B = using "semi-local" basis functions.

h	Re = 0 A	Re = 0 B	Re = 1 A	Re = 1 B	Re = 10 A	Re = 10 B
1/4	-.417	-.494	-.418	-.495	-.429	-.507
1/8	-.466	-.496	-.468	-.497	-.487	-.503
1/16	-.482		-.484	-.498	-.501	-.500
1/4	-.214	-.218	-.216	-.218	-.235	-.224
1/8	-.235	-.260	-.238	-.233	-.268	-.241
1/16	-.250		-.253	-.238	-.281	-.245
1/5	.372	.360	.371	.359	.348	.353
1/10	.377	.371	.374	.371	.347	.368
1/15	.378		.375	.373	.347	

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Finite element algorithms are presented for the approximate solution of the streamfunction-vorticity equations of steady incompressible viscous flows. Both the linear Stokes and the nonlinear Navier-Stokes equations are considered. The methods discussed require low continuity finite element spaces and do not require any artificial specification of the vorticity at solid boundaries. Particular attention is paid to methods for multiply connected domains and to theoretical and computational estimates for the accuracy of the algorithms. Brief consideration is also given to three dimensional problems, to exterior problems, and to the recovery of the pressure field.

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